INTEGRAL SIMILAR TO GAMMA FUNCTION

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1. Introduction: Mathematics plays a vital role in developing creativity and beautiful connection with nature. Through mathematics many new doors have been open for science for research .One such door has been opened by integrals. Integral Calculus is one of the most important branch of mathematics which have given us information about infinite series , and some of the most beautiful functions like "Gamma function ".

2. Abstract:
   - The overall purpose of study and research.

Mathematics acts as a universal language which allows various intellectuals to communicate and study the different patterns in universe. Without mathematics we can't even imagine how this world would have survived. Without mathematics subjects like Physics would not be completed as there would have been no proof to prove theories.Inshort it provides platform to various other fields of science. So it's very necessary to contribute in mathematics.

3. Higher education in mathematics:

Maths plays a vital role in calculations and analysis of scientific theories. Not only in theory but in engineering also like in automobile engineering. Mathematics is itself an art and which is gifted by god, and I think our life has become easy due to mathematics. Imagine if there were no numbers how could computers would have been created. So for survival on this planet we need to have interest in maths so to serve these world new interventions. So we should opt mathematics as a subject in higher studies like graduation so that we can one day become good mathematicians and develop new ideas and investigate this universe.

4. Famous infinite series which changed the world : Since we know whenever we talk about infinite series in mathematics, we come with one name which was one of the greatest mathematician of all time "S. Ramanujan."He gave a whole new concept to understand divergent series like "sum of all natural numbers up to infinity." Isn't it mindboggling? He even assigned value to this series which was "-1/12".

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5. <u>The basic layout of my study</u> :

In my study I have developed an integral which opens new paths of research in mathematics. It provides a whole greater view to understand infinite series which seem difficult to understand without this integral.

$$P(n) = \int_0^\infty e^{-x}.(\sin x)^{n-1}dx$$

With this beautiful integral i was able to assign values to series like:

$$\left[2^{\circ}\left(4^2-4\right)-2^2\left(4^3-4\right)+2^4\left(4^4-4\right)-........\right]=\frac{12}{85}$$

$$\text{Since gamma (n)} = \int_0^\infty e^{-x}.x^{n-1}dx$$

This gamma function provides vital knowledge in mathematics. So I have devolved relation in my integral p(n) and gamma function.

So i think it is very necessary to be published in textbooks so that students should get ideas to understand series in better way.

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Consider, P(n) = $\int_0^\infty e^{-x}.(\sin x)^{n-1}dx$

(by using I LATE rule of integration)

$= \left[-(\sin x)^{n-1}e^{-x}\right]^\infty + \int_0^\infty (n-1)(\sin x)^{n-1}\cos x e^{-x}dx$

= [0 – 0] + (n – 1) $\int_0^\infty (\sin x)^{n-2}(\cos x)e^{-x}dx$

= (n – 1) $\int_0^\infty (\sin x)^{n-2}(\cos x)e^{-x}dx$ ------- (1)

$\Rightarrow$ now solving above integral $\int_0^\infty (\sin x)^{n-2}(\cos x)e^{-x}dx$

$\Rightarrow \int_0^\infty (\sin x)^{n-2}\cos e^{-x}dx$ ------- (2)

(again using I LATE rule of integration)

$= \left[-(\sin)^{n-2}\cos x e^{-x}\right]_0^\infty + \int_0^\infty \left[(n-2)(\sin x)^{n-3}\cos^2 x\cos^2 x-(\sin x)^{n-1}\right]e^{-x}dx$

= [0 – 0] + (n – 2) $\int_0^\infty e^{-x}(\sin x)^{n-3}\cos^2 xdx$ – $\int_0^\infty e^{-x}(\sin x)^{n-1}dx$

= (n – 2) $\int_0^\infty e^{-x}(\sin x)^{n-2}\left[1-\sin^2 x\right]dx$ – $\int_0^\infty e^{-x}(\sin x)^{n-1}$ dx

= (n – 2) $\left[\int_0^\infty e^{-x}(\sin x)^{n-3}dx-\int_0^\infty e^{-x}(\sin x)^{n-1}dx\right]$ – $\int_0^\infty e^{-x}..(\sin x)^{n-1}dx$

$$\left[\begin{array}{l}\sin ceP(n)=\int_0^\infty e^{-x}.(\sin x)^{n-1}dx\\ \therefore P(n-2)=\int_0^\infty e^{-x}.(\sin x)^{n-3}dx\end{array}\right]$$

= (n – 2) [P(n – 2) – P (n)] – P(n)

So we get,

$$\int_0^\infty e^{-x}(\sin x)^{n-2}\cos xdx=(n-2)\left[P(n-2)-p(n)\right]-P(n)$$

Now put this value of integral in (1), we get

$P(n) = (n - 1)\ [(n-2)[P(n-2) - P(n)] - P(n)]$

$= (n - 1)\ [(n-2)P(n-2) - (n-2)P(n) - P(n)]$

$= (n - 1)\ [(n-2)P(n-2) - P(n)(n-1)]$

$\therefore\ P(n) = (n - 1)\ (n - 2)\ P\ (n - 2) - P\ (n)\ (n - 1)^2$

$P(n)\ [1 + (n - 1)^2] = (n - 1)\ (n - 2)\ P\ (n - 2)$

So we get,

$P(n)\ -\ \dfrac{(n-1)(n-2)P(n-2)}{1+(n-1)^2}$ [where n > 2]

Now For n = 1

$P\ (1) = \int_0^{\infty} e^{-x}.(\sin x)^{\circ} dx$

$= \int_0^{\infty} e^{-x}.dx = 1$

For n = 2, $P(2) = \int_0^{\infty} e^{-x}.(\sin x)\ dx$

$= [-\sin x\ e^{-x}]^{\infty}_0 + \int_0^{\infty} e^{-x} \cos x dx$

$= 0 + \int_0^{\infty} e^{-x} \cos x dx$

$= [-\cos x\ \ e^{-x}]^{\infty}_0 - \int_0^{\infty} \sin x e^{-x} dx$

$= 1 - P\ (2)$

$\therefore\ \ P(2) = 1 - P\ (2)$

$2P(2) = 1$

$\therefore\ P(2) = \dfrac{1}{2}$

For n = 0 $\qquad P(0) = \int_0^{\infty} e^{-x}.(\sin x)^{-1} dx$

$= \int_0^{\infty} e^{-x}.\cos ex \qquad dx$

= [- cosex e $^{-x}$]$^{\infty}_{0}$ - $\int_0^\infty \cos ex \cot x e^{-x} dx$

Since [- cos ex e $^{-x}$]$^{\infty}_{0}$ is undefined

So for n = 0, it is undefined.

⇒ Now consider P(n) = $\int_0^\infty e^{-x}.(\sin x)^{n-1} dx$

If n – 1 < 0

n < 1

consider n = 1 – m

(where m > 0 any positive number)

∴ P (1 – m) = $\int_0^\infty e^{-x}.(\sin x)^{-m} dx$

= $\int_0^\infty e^{-x}.(\cos ex)^{m} dx$

= [– (cos ex)^{m} e $^{-x}$]$^{\infty}_{0}$ – (m) $\int_0^\infty (\cos ex)^{m-1} \cos ex \cot x e^{-1} dx$

Now [- (e) ^-x (cos ex) ^m]$^{\infty}_{0}$ = $\left[\frac{-e^{-x}}{(\sin x)^m}\right]_0^\infty$ is undefined.

So P (1 – m) is also undefined.

So, together P(n) = $\int_0^\infty e^{-x}.(\sin x)^{n-1} dx$

has defined value for n ? 0,where n ∈ N.

⇒ now further consider for P (n)

Since P(n) = $\frac{(n-1)(n-2)P(n-2)}{\left[1+(n-1)^2\right]}$ → 3

P (n – 2) = $\frac{(n-3)(n-4)P(n-4)}{\left[1+(n-3)\right]^2}$ → 4

P (n – 4) = $\frac{(n-5)(n-6)P(n-6)}{\left[1+(n-5)\right]^2}$

so on …….

Case (I) when n is odd

$$\Rightarrow P(n) = \frac{(n-1)(n-2)\cancel{P(n-2)}}{\left[1+(n-1)^2\right]}$$

$$\cancel{P(n-2)} = \frac{(n-3)(n-4)\cancel{P(n-4)}}{\left[1+(n-3)^2\right]}$$

$$\cancel{P(n-4)} = \frac{(n-5)(n-6)\cancel{P(n-6)}}{\left[1+(n-5)^2\right]}$$

Multiply & do cutting in both,

we get.

$$P(3) = \frac{(2)(1)P(1)}{\left[1+2^2\right]}$$

$$\Rightarrow P(n) = \frac{(n-1)(n-2)(n-3)(n-4)\ldots\ldots.(2)(1)P(1)}{\left[1+(n-1)^2\left[1+(n)-3\right]^2\left[1+(n-5)^2\right]\ldots\ldots.\left[1+2^2\right]\right.}$$

$$\Rightarrow P(n) = \frac{\angle n+1}{\left[1+(n-1)^2\left[1+(n-3)^2\left[1+(n-5)\right]^2\ldots\ldots\ldots.\left[1+2^2\right]\right]\right.}$$

Since n is odd, consider n = 2k + 1

Where $k \in N$

$$\Rightarrow P(2k+1) = \frac{\angle 2K}{\left[1+(2k)^2\left[1+(2k-2)^2\left[1+(2k-4)^2\right]\ldots\ldots\left[1+2^2\right]\right.\right.}$$

$$P(2k+1) = \frac{\angle 2K}{\left[1+2^2k^2\left[1+2^2(k-1)^2\left[1+2^2(k-2)^2\right]\ldots\ldots\left[1+2^2-1^2\right]\right.\right.}$$

$$\Rightarrow P(2K+1) = \frac{\angle 2K}{r=k-1\left[1+2^2(k-r)^2\right]}$$
$r = 0$

$$\Rightarrow P(2K+1) = \frac{2k+1}{r=k-1\left[1+2^2(k-r)^2\right]}$$
$r = 0$

Case (II) when n is even

$$\Rightarrow P(n) = \frac{(n-1)(n-2)\cancel{P(n-2)}}{\left[1+(n-1)^2\right]}$$

$$\Rightarrow \cancel{P(n-2)} = \frac{(n-3)(n-4)\cancel{P(n-4)}}{\left[1+(n-3)^2\right]}$$

$$\Rightarrow \cancel{P(n-4)} = \frac{(n-5)(n-6)\cancel{P(n-6)}}{\left[1+(n-5)^2\right]}$$

$$P(4) = \frac{(3)(2)P(2)}{\left[1+3^2\right]} \qquad \text{since } P(2) = \frac{1}{2}$$

$$\Rightarrow P(n) = \frac{(n-1)(n-2)(n-3)(n-4).......(2)(1)P(L)}{\left[1+(n-1)^2\right]\left[1+(n-3)^2\right]\left[1+(n-s)^2\right].......\left[1+3^2\right]}$$

Since $P(2) = \frac{1}{2}$

$$P(n) = \frac{n-1.\frac{1}{2}}{\left[1+(n-2)^2\right]\left[1+(n-3)^2.......(1+3^2)\right]}$$

Since n is even, put n = 2k + 2 when k ∈ I N

$$\Rightarrow P(2k+2) = \frac{\angle 2k\ +1.\ \frac{1}{1}}{\left[1+(2k+1)^2\right]\left[1+(2k-1)^2\right]..........\left[1+3^2\right]}$$

$$\Rightarrow P(2k+2)\ \frac{\angle 2k+1.\frac{1}{2}}{\underset{r=1}{r=k}\left[1+(2(k-r)+3)^2\right]}$$

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So that end we get relation

$$\text{P}\ (2k + 2) = \frac{2k+2.\frac{1}{2}}{\underset{r-1}{r=k}\left[1+\left(2(k-r)+3\right)^{2}\right]}$$

So we get relation between Gamma and P(n) function, in both even and odd cases.

Now at the end we have

$$\text{P}(n) = \int_{0}^{\infty} e^{-x}.(\sin x)^{n-1}\,dx$$

Where P(1) = 1

$$\text{P}(2) = \frac{1}{2}$$

Case (I) when n is odd where n = 2k + 1, k $\in$ N

$$\text{P}(2k+1) = \frac{2k+1}{\underset{r=0}{r=k-1}\left[1+2^{2}(k-r)^{2}\right]}$$

Case (II) when n is even, where n = 2k + 2, k $\in$ N

$$\text{P}(2k + 2) = \frac{2k+2.\frac{1}{2}}{\underset{r=1}{r=k}\left[1+\left(2(k-r)+3\right)^{2}\right]}$$

So by using the above integral we can have formulate various series line:

i) $\left[2^{\circ}\left(4^{2}-4\right)-2^{2}\left(4^{3}-4\right)+2^{4}\left(4^{4}-4\right)-........\right]=\frac{12}{85}$

ii) $\left[\left(3^{3}-5^{2}-2\right)-\left(3^{5}-5^{4}-2\right)+\left(3^{7}-5^{6}-2\right)-.........\right]=\frac{48}{65}$

Now prove of the above series:

i) $\left[2^{\circ}\left(4^{2}-4\right)-2^{2}\left(4^{3}-4\right)+2^{4}\left(4^{4}-4\right)-........\right]=\frac{12}{85}$

Consider P(5) = $\int_0^\infty e^{-x}.\sin^4 x dx$

= $\int_0^\infty e^{-x}.(\sin^2 x)^2 dx$

Now we will write $\sin^4 x$ in term of sin & cos.

Sin4x = $(\sin^2 x)^2$

= $\left(\frac{1-\cos 2x}{2}\right)^2$ = $\frac{1+\cos^2 2x - 2\cos 2x}{4}$

$$\left[\begin{array}{l} \sin ce \quad \cos^4 x = 2\cos^2 2x - 1 \\ \frac{\cos 4x + 1}{2} = \cos^2 2x \end{array}\right]$$

$\sin^4 x$ = $\dfrac{1 + \frac{1+\cos 4x}{2} - 2\cos^2 x}{4}$

= $\frac{1}{4}\left[1 + \frac{1}{2} + \frac{\cos 4x}{2} - 2\cos 2x\right]$

$\Rightarrow$ $\sin^4$x = $\frac{1}{8}[3 + \cos 4x - 4\cos 2x]$

$\Rightarrow$ P(5) = $\int_0^\infty e^{-x}.\frac{1}{8}[3 + \cos 4x - 4\cos 2x]dx$

= $\frac{1}{8}\int_0^\infty e^{-x}.[3 + \cos 4x - 4\cos 2x]dx$

= $\frac{1}{8}\left[3\int_0^\infty e^{-x} dx + \int_0^\infty e^{-x} \cos 4x dx - 4\int_0^\infty e^{-x}.\cos 2x dx\right]$

= $\frac{1}{8}\left[3\int_0^\infty e^{-x} dx + \int_0^\infty e^{-x}[\cos 4x - 4\cos 2x]dx\right]$

Now expanding term [cos 4x – 4cos 2x], we get $\Rightarrow$

Cos4x – 4cos 2x

$$= \begin{bmatrix} \left(1 - \frac{(4x)^2}{\angle 2} + \frac{(4x)^4}{\angle 4} - \frac{(4x)^6}{\angle 6} + \ldots\ldots\right) \\ + 4\left(-1 + \frac{(2x)^2}{\angle 2} - \frac{(2x)^4}{\angle 4} + \frac{(2x)^6}{\angle 6} - \ldots\ldots\ldots\ldots\right) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \left(\cancel{1} - \frac{(4x)^2}{\angle 2} + \frac{(4x)^4}{\angle 4} - \frac{(4x)^6}{\angle 6} + \ldots\ldots\right) \\ + \left(-\cancel{4} + \cancel{\frac{4(2x)^2}{\angle 2}} + \frac{4(2x)^4}{\angle 4} - \frac{4(2x)^6}{\angle 6} - \ldots\ldots\ldots\ldots\right) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{(2x)^4}{\angle 4}\left(-4+4^2\right) + \frac{(2x)^6}{\angle 6}\left(4-4^3\right) + \frac{(2x)^8}{\angle 8}\left(-4+4^4\right) + \ldots\ldots\ldots \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{(2x)^4}{\angle 4}\left(-4+4^2\right) - \frac{(2x)^6}{\angle 6}\left(-4+4^3\right) + \frac{(2x)^8}{\angle 8}\left(-4+4^4\right) - \ldots\ldots\ldots\ldots \\ -3 \end{bmatrix}$$

So we get, since $\int_0^\infty e^{-x}dx = 1$

$$= \frac{1}{8}\left[\cancel{3} + \int_0^\infty e^{-x}\left[\left(\frac{(2x)^4}{\angle 4}\left(-4+4^2\right) - \frac{(2x)^6}{\angle 6}\left(-4+4^3\right) + \frac{(2x)^8}{\angle 8}\left(-4+4^4\right) + \ldots\ldots\right) - \cancel{3}\right]dx\right]$$

$$= \frac{1}{8}\left[\int_0^\infty e^{-x}\left[\frac{(2x)^4}{\angle 4}\left(-4+4^2\right) - \frac{(2x)^6}{\angle 6}\left(-4+4^3\right) + \frac{(2x)^8}{\angle 8}\left(-4+4^4\right) - \ldots\ldots\right]dx\right]$$

Since gamma (n) $= \int_0^\infty e^{-x}.x^{n-1}dx$

Using this we get

$$= \frac{1}{8}\left[\frac{2^4\left(-4+4^2\right)}{\cancel{\angle 4}}.\ \cancel{\lceil 5} - \frac{(2)^6}{\cancel{\angle 6}}\left(-4+4^3\right).\ \cancel{\lceil 7} + \frac{(2)^8}{\cancel{\angle 8}}\left(-4+4^4\right).\ \cancel{\lceil 9}\ldots\ldots\ldots\right]$$

$$= \frac{1}{8}\left[2^4\left(-4+4^2\right) - 2^6\left(-4+4^3\right) + 2^8\left(-4+4^4\right)\ldots\ldots\ldots\right]$$

$$=\frac{2^4}{8}\left[\left(4^2-4\right)-2^2\left(4^3-4\right)+2^4\left(4^4-4\right)-.........\right]$$

$$=\frac{{}^{2}\cancel{16}}{\cancel{8}}\left[\left(4^2-4\right)-2^2\left(4^3-4\right)\right]+2^4\left(4^4-4\right)-.......]$$

$$=2\left[2^{\circ}\left(4^2-4\right)2^2\left(4^3-4\right)+2^4\left(4^4-4\right)-.........\right] = \text{P (5)}$$

Since 5 is odd using formula

$$\text{P(2k + 1)} = \frac{\sqrt{2k+1}}{\underset{r=0}{r=k-1}\left[1+2^2\left(k-r\right)^2\right]}$$

Put k =0, we get

$$\text{P(5)} = \frac{\sqrt{5}}{\underset{r=0}{r=1}\left[1+2^2\left(2-r\right)^2\right]}$$

$$\Rightarrow \text{P(5)} = \frac{\sqrt{5}}{\left[1+2^2\left(2-0\right)^2\right]\left[1+2^2\left(2-1\right)^2\right]}$$

$$=\frac{\angle 4}{\left[1+16\right]\left[1+4\right]}$$

$$=\frac{(4)(3)(2)}{(17)(5)}=\frac{24}{85}$$

Similar for P(6)= $\int_0^\infty e^{-x}.\sin^5 x dx$

We get series:

$$\left[\left(3^{3-}5^2-2\right)-\left(3^5-5^4-2\right)+\left(3^7-5^6-2\right)-.........\right]= \frac{48}{65}$$